

From the table, it is seen that $f' = G = 0.99$ for $\eta = 5$. So, u/U_∞ reaches 0.99 at $\eta = 5$. It is possible to write

$$\frac{\delta}{\sqrt{\frac{\nu x}{U_\infty}}} = 5; \quad \text{or} \quad \frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}$$

2.2 TEMPERATURE DISTRIBUTION OVER FLAT PLATE BOUNDARY LAYER

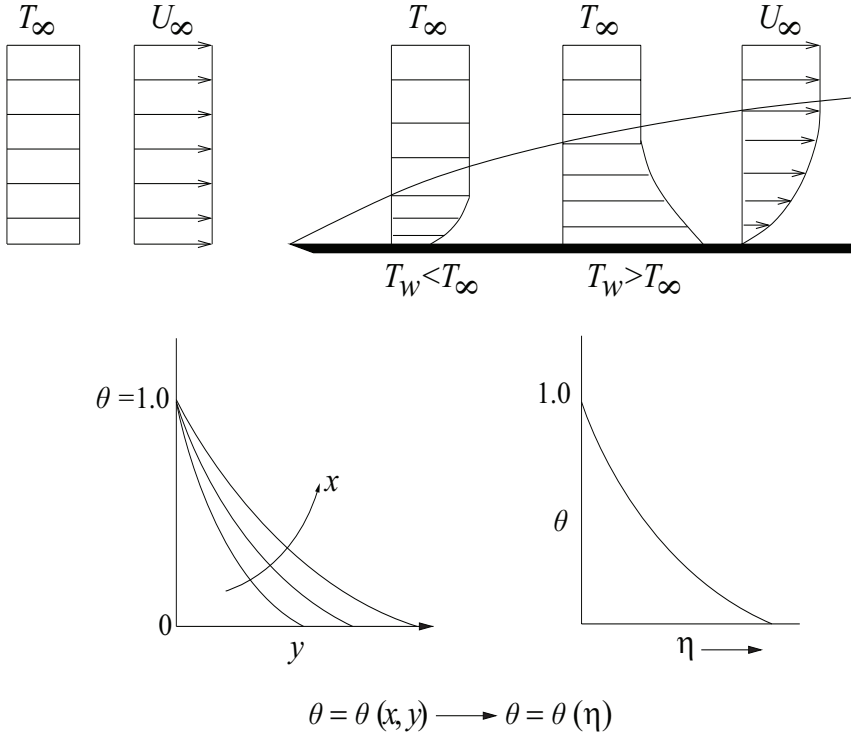


Figure 2.3 The growth of thermal boundary layer

Thermal boundary layer equation

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \quad (2.30)$$

$$u = U_\infty f'(\eta), v = -\frac{1}{2} \sqrt{\frac{\nu U_\infty}{x}} \{f(\eta) - \eta f'(\eta)\} \quad (2.31)$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \quad (2.32)$$

Figure 2.3 describes the growth of thermal boundary layer on a heated flat plate.

The boundary condition for the situation described above is $\theta(0) = 1$, $\theta(\infty) = 0$. Therefore, $T = T_\infty + \theta(T_w - T_\infty)$.

Let us assume $T_w = T_w(x)$ (a general case). On substituting $\eta = (y/\sqrt{\frac{\nu x}{U_\infty}})$, we can write different derivatives as

$$\frac{\partial T}{\partial x} = \theta \frac{dT_w}{dx} + (T_w - T_\infty) \left(-\frac{y}{2x}\right) \sqrt{\frac{U_\infty}{\nu x}} \frac{d\theta}{d\eta} \quad (2.33)$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial y} = (T_w - T_\infty) \frac{\partial \theta}{\partial \eta} \sqrt{\frac{U_\infty}{\nu x}} \quad (2.34)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial \eta} \left[\frac{\partial T}{\partial \eta} \right] \frac{\partial \eta}{\partial y} = (T_w - T_\infty) \frac{\partial^2 \theta}{\partial \eta^2} \frac{U_\infty}{\nu x} \quad (2.35)$$

On substituting the derivatives in thermal boundary layer equation and multiplying by $\frac{x}{U_\infty(T_w - T_\infty)}$ on both sides of the equation, we get

$$\frac{f' \theta x}{(T_w - T_\infty)} \frac{dT_w}{dx} - \frac{f' y}{2} \sqrt{\frac{U_\infty}{\nu x}} \theta' - \frac{(f - \eta f')}{2} \theta' = \frac{k}{\rho c_p} \frac{\theta''}{\nu} \quad (2.36)$$

$$f' \theta x \frac{d}{dx} [\ln(T_w - T_\infty)] - \frac{f' \eta \theta'}{2} - \frac{(f - \eta f') \theta'}{2} = \frac{k}{\mu c_p} \theta'' \quad (2.37)$$

Dividing by $f' \theta$

$$x \frac{d}{dx} [\ln(T_w - T_\infty)] - \frac{f \theta'}{2 f' \theta} = \frac{k}{\mu c_p} \theta'' \frac{1}{f' \theta} \quad (2.38)$$

This equation could be written in the form

$$\frac{x(dT_w/dx)}{T_w(x) - T_\infty} = \frac{k \theta''(\eta)}{\mu c_p f'(\eta) \theta(\eta)} + \frac{f(\eta) \theta'(\eta)}{2 f'(\eta) \theta(\eta)} = \lambda \quad (2.39)$$

This equation can be solved by using the method of separation of variables. This leads to

$$\frac{x(dT_w/dx)}{T_w(x) - T_\infty} = \lambda \quad (2.40)$$

Wall temperature has to follow this relation

$$d \{ \ln(T_w - T_\infty) \} = \frac{dx}{x} \lambda \quad (2.41)$$

Integrating we get,

$$\ln(T_w - T_\infty) = \lambda \ln x + C \quad (2.42)$$

or

$$(T_w - T_\infty) = Cx^\lambda$$

when

$$\lambda = 0, T_w = T_\infty + C \quad (\text{constant wall temperature})$$

Again,

$$\frac{k\theta''(\eta)}{\mu c_p f'(\eta)\theta(\eta)} + \frac{f(\eta)\theta'(\eta)}{2f'(\eta)\theta(\eta)} = \lambda \quad (2.43)$$

or

$$\frac{\theta''(\eta)}{Pr} + \frac{1}{2}f\theta' = \lambda f'\theta \quad (2.44)$$

or

$$\theta'' + \frac{Pr}{2}f\theta' = \lambda Pr f'\theta \quad (2.45)$$

$$\theta'' + \frac{Pr}{2}f\theta' - \lambda Pr f'\theta = 0 \quad (2.46)$$

In this equation f is known from the Blasius solution. The boundary conditions are:

$$y = 0, T = T_w \rightarrow \eta = 0, \theta = 1$$

$$y = \infty, T = T_\infty \rightarrow \eta = \infty, \theta = 0$$

$$T = T_\infty + \theta(T_w - T_\infty)$$

The final forms of the boundary conditions are $\theta(0) = 1$ and $\theta(\infty) = 0$.

Let $Y_1 = \theta$ and $Y_2 = \theta'$. Equation (2.46) can be written as

$$Y_2' = -\frac{Pr}{2}fY_2 + \lambda Pr f'Y_1 \quad (2.47)$$

with $Y_1(0) = 1$ and $Y_1(\infty) = 0$.

The numerical solution using Runge-Kutta method can be obtained via following steps

1. Estimate a value of $Y_2(0)$ or $\theta'(0)$.
2. Solve for Y_1, Y_2 .
3. Check if $Y_1(\infty) = 0$, i.e., $\theta(\infty) = 0$?
4. If yes, stop. The calculated $\theta(\eta)$'s are correct solution.

5. If no, correct $\theta'(0)$ or $Y_2(0)$ and repeat the calculation.

The Equation (2.16) and Equation (2.47) can be solved together numerically for obtaining the similarity profiles for velocity and temperature. As already mentioned, basic Runge-Kutta procedure is deployed for obtaining the solutions. A computer code, developed for this purpose, has been provided in the appendix at the end of this chapter.

2.3 ANALYTICAL SOLUTION FOR $\lambda = 0$

$$\theta'' + \frac{Pr}{2} f \theta' = 0 \quad (2.48)$$

or

$$\frac{\theta''}{\theta'} = -\frac{Pr}{2} f \quad (2.49)$$

or

$$\frac{d}{d\eta}(\ln \theta') = -\frac{Pr}{2} f$$

or

$$\ln \theta' = -\frac{Pr}{2} \int f(\eta) d\eta + \ln C$$

or

$$\theta' = C e^{-\frac{Pr}{2} \int f(\eta) d\eta}$$

or

$$\theta = C \int e^{-\frac{Pr}{2} \int f(\eta) d\eta} d\eta + D$$

or

$$\theta(\eta) - \theta(0) = C \int_0^\eta e^{-\frac{Pr}{2} \int_0^\eta f(\eta) d\eta} d\eta$$

or

$$\theta(\eta) = 1 + C \int_0^\beta (e^{-\frac{Pr}{2} \int_0^\beta f(r) dr}) d\beta \quad (2.50)$$

Now $\theta(\infty) = 0$ will result in

$$-1 = C \int_0^\infty (e^{-\frac{Pr}{2} \int_0^\beta f(r) dr}) d\beta \quad (2.51)$$

or

$$C = -\frac{1}{\int_0^\infty (e^{-\frac{Pr}{2} \int_0^\beta f(r) dr}) d\beta} \quad (2.52)$$

$$\theta(\eta) = 1 - \frac{\int_0^\eta (e^{-\frac{Pr}{2} \int_0^\beta f(r) dr}) d\beta}{\int_0^\infty (e^{-\frac{Pr}{2} \int_0^\beta f(r) dr}) d\beta}$$

$$\theta(\eta) = 1 - \frac{\int_0^\eta (e^{-\frac{Pr}{2}} \int_0^\eta f(\eta) d\eta) d\eta}{\int_0^\infty (e^{-\frac{Pr}{2}} \int_0^\eta f(\eta) d\eta) d\eta} \quad (2.53)$$

Nu_x = Nusselt number = $\frac{hx}{k}$

$$h = \frac{-k(dT/dy)_{y=0}}{(T_w - T_\infty)} = \frac{-k \left[\frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial y} \right]_{\eta=0}}{(T_w - T_\infty)} \quad (2.54)$$

We know,

$$T = T_\infty + \theta (T_w - T_\infty) \text{ we can write, } \frac{\partial T}{\partial \eta} = \frac{\partial \theta}{\partial \eta} (T_w - T_\infty)$$

$$h = -k \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial y} \Big|_{\eta=0} = -k \theta'(0) \sqrt{\frac{U_\infty}{\nu x}} \quad (2.55)$$

$$Nu_x = \frac{hx}{k} = -\theta'(0) \sqrt{\frac{U_\infty x}{\nu}} = -\theta'(0) (Re_x)^{1/2} \quad (2.56)$$

$$\frac{Nu_x}{(Re_x)^{1/2}} = -\theta'(0) \simeq 0.332 Pr^{1/3} \quad (0.6 \leq Pr \leq 10) \quad (2.57)$$

The solution was obtained by Pohlhausen [4].

2.4 MORE DISCUSSIONS ON SOLUTION VIA NUMERICAL ROUTE

The heat transfer rate at the wall is given by

$$q_w'' = -k \frac{\partial T}{\partial y} \Big|_{y=0} \quad (2.58)$$

Hence,

$$\frac{q_w''}{k(T_w - T_\infty)} = -\frac{\partial \theta}{\partial \eta} \Big|_{\eta=0} = -\frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial y} \Big|_{y=0} \quad (2.59)$$

i.e.,

$$\frac{q_w'' x}{k(T_w - T_\infty)} = -\theta' \Big|_{\eta=0} \sqrt{Re_x} \quad (2.60)$$

i.e.,

$$Nu_x = -\theta' \Big|_{\eta=0} \sqrt{Re_x} \quad (2.61)$$

Nu_x and Re_x , are, the local Nusselt and Reynolds numbers based on x . Because θ depends only on η for a given Pr , $\theta' \Big|_{\eta=0}$ depends only on Pr and its values can be obtained from the solution for the variation θ with η for any value of Pr . It is convenient to define.

$$A(Pr) = -\theta' \Big|_{\eta=0} \quad (2.62)$$

In terms of this function A, Eq. (2.61) can be written as

$$Nu_x = A \sqrt{Re_x} \quad (2.63)$$

Values of A for various values of Pr are shown in Table 2.2.

Table 2.2
Values of A for various values of Pr

Pr	$A = -\theta' _{\eta=0}$	$0.332Pr^{1/3}$
0.6	0.276	0.280
0.7	0.293	0.295
0.8	0.307	0.308
0.9	0.320	0.321
1.0	0.332	0.332
1.1	0.344	0.343
7.0	0.645	0.635

Over the range of Prandtl numbers covered in the table, it can be found that A varies very nearly as $Pr^{1/3}$ and, as will be seen from the results given in the table, is quite closely represented by the approximate relation:

$$A = 0.332Pr^{1/3} \quad (2.64)$$

We calculate the total heat transfer rate from a plate of length, L . Unit width of the plate is considered because the flow is, by assumption, two-dimensional. The total heat transfer rate per unit width Q_w will, of course, be related to the local heat transfer rate, q_w'' by:

$$Q_w = \int_0^L q_w'' dx \quad (2.65)$$

But Eq. (2.60) gives the local heat transfer rate as

$$q_w'' = Ak(T_w - T_\infty) \sqrt{\frac{U_\infty}{x\nu}} \quad (2.66)$$

Substituting this result Eq. (2.65) then continues the integration:

$$Q_w = 2Ak(T_w - T_\infty) \sqrt{\frac{UL}{\nu}} \quad (2.67)$$

If a mean heat transfer coefficient for the whole plate is defined such that

$$\bar{h} = \frac{Q_w}{L(T_w - T_\infty)} \quad (2.68)$$

Then, since unit width of the plate is being considered, Eq. (2.67) gives

$$\bar{h} = \frac{2Ak}{L} \sqrt{\frac{U_\infty L}{\nu}}$$

It is possible to calculate average Nusselt number as

$$\overline{Nu} = \frac{\bar{h}L}{k} = 2A\sqrt{Re_L} \quad (2.69)$$

Let us revisit the thermal conditions for the wall again. We consider the case where:

$$T_w = T_\infty + Cx^\lambda$$

i.e.,

$$T_w - T_\infty = Cx^\lambda \quad (2.70)$$

C and λ being constants. The heat transfer rate at the wall is as before given by

$$q_w'' = -k \frac{\partial T}{\partial y} \Big|_{y=0} \quad (2.71)$$

$$q_w'' = -k(T_w - T_\infty) \frac{\partial \theta}{\partial \eta} \Big|_{\eta=0} \frac{\partial \eta}{\partial y}$$

$$q_w'' = -kC x^\lambda \sqrt{\frac{U_\infty}{\nu x}} \frac{\partial \theta}{\partial \eta} \Big|_{\eta=0}$$

$$q_w'' = \frac{-kC\sqrt{U_\infty}}{\sqrt{\nu}} \theta'(0) x^{\lambda-1/2} \quad (2.72)$$

For the constant heat flux along the wall, the expression should be independent of x or in other words, we can say

$$\lambda = \frac{1}{2} \quad (2.73)$$

2.5 APPROXIMATE METHODS FOR FLAT PLATE BOUNDARY LAYER

Flow over a heated flat plate for different Prandtl numbers $\left(\frac{\delta}{\delta_T} \sim (Pr)^{1/3} \text{ for } Pr > 1 \text{ or } \frac{\delta}{\delta_T} \sim (Pr)^{1/2} \text{ for } Pr < 1\right)$ has been illustrated in Fig. 2.4. Governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.74)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad (2.75)$$